NOTE: Add the figures for product rule and 1-D chain rule!!

1 Zapping with d

Suppose you have a symbolic equation for a state variable (an "equation of state", and want to find its differential. How do you manage that? We call our approach "zap the equation with d".

As a first example, let's consider a (relatively) simple equation:

$$\sin\left(\frac{x_L + x_R}{a}\right) = \log\left(\frac{F_L}{F_R}\right) \tag{1}$$

$$= \log(F_L) - \log(F_R) \tag{2}$$

You could solve this equation for x_L and then find a derivative, but then you'd have to deal with arcsines and all sorts of nastiness. It is easier to just zap both sides of the equation with d. Zapping with a d is like taking a derivative (with the same chain rule and product rule), except that you change everything that *can* change. In the above example, this comes down to

$$\cos\left(\frac{x_L + x_R}{a}\right)\left(\frac{dx_L}{a} + \frac{dx_R}{a}\right) = \frac{dF_L}{F_L} - \frac{dF_R}{F_R} \tag{3}$$

$$dx_L = \frac{a}{\cos\left(\frac{x_L + x_R}{a}\right)}\left(\frac{dF_L}{F_L} - \frac{dF_R}{F_R}\right) - dx_R \tag{4}$$

where a is a constant. Note that our expression for the differential dx_L itself involves x_L. We can, of course, eliminate x_L from this equation, but you do not need do that unless it is convenient (or needed for some further step).

So how did we actually zap this with d? How can you learn to do the same? Read on.

1.1 Product rule

Ordinary symbolic differentiation generally comes down to two rules, plus remembering the derivatives of functions such as sine, cosine, and powers. The first rule is the product rule. When zapping with d, this rule is compactly expressed as:

$$f = ab \tag{5}$$

$$df = a\,db + b\,da \tag{6}$$

In other words, the differential of a product is the sum of the differentials of each factor times the remaining factor. Even if you don't remember the derivation of the product rule, this is pretty easy to see geometrically, if you remember that the product of two numbers gives the area of a rectangle, as in Figure

1.2 1-D Chain rule

The product rule told us how to zap two things multiplied together. The chain rule tells us how to zap a function of something else.

$$f = g(a) \tag{7}$$
$$df = g'(a)da \tag{8}$$

we have assumed here that the function g is a single-variable function, since all of our common functions are indeed functions of a single variable. we also used the "prime" notation, so as to avoid the need to give a name to the argument of g. Figure

It is worth noting that the chain rule expressed in this way looks very much like the definition of a derivative. The difference is that now we recognize da itself as not only "a small change in a", but possibly also some more complicated total differential, so that for instance (skipping steps) we can recognize that if x and t are variables, while k and ω are constants, then

$$f = e^{i(kx-\omega t)} \tag{9}$$
$$df = e^{i(kx-\omega t)}\left(ik\,dx - i\omega\,dt\right) \tag{10}$$

By the way, you may have been taught a quotient rule as well. we would recommend forgetting it for the moment, and just remembering that $b/a = b(a)^{-1}$ and using the product rule and the chain rule.

1.3 Putting the rules together

In defining the product and chain rule, we used the expressions a and b. In reading this, you probably thought of these as variables, but they could actually be any arbitrary expression! One of the beautiful things about zapping with d is that you can define as many convenience variables as you like without adding any complexity. Let me give an example. Consider the function

$$f = \sin\left(\frac{x_1 x_2}{a^2}\right) \tag{11}$$

This is the sine of another function. We can give names to everything, to make things as simple as possible.

$$f = \sin b \tag{12}$$
$$b = \frac{c}{a^2} \tag{13}$$
$$c = x_1 x_2 \tag{14}$$

Now we can zap each equation with d!

$$df = \cos b\,db \tag{15}$$
$$db = \frac{dc}{a^2} \tag{16}$$
$$dc = x_1 dx_2 + x_2 dx_1 \tag{17}$$

where we assumed that a was a constant. If it weren't a constant, then we would change the second equation to

$$db = \frac{dc}{a^2} - 2\frac{c}{a^3}da \tag{18}$$

and you might have wished to give a name to a^{-2} to help with this step.

Philosophically, we think of "taking derivatives" as acting on functions, each of which must have a known set of independent variables. In contrast, "zapping with d" acts on state variables, which need not have a specified set of independent variables as input. The result of zapping with d is in the general case an equation relating differentials that may include some redundancies, due to having more differentials than there are independent degrees of freedom. That is all right, the equation is still true!

For more information on this topic, you can see the following link, which discusses how to organize your process using chain rule diagrams: `https://paradigms.oregonstate.eduhttp://math.oregonstate.edu/BridgeBook/book/math/chainddiag`