

Find the upward pointing flux of the electric field  $\vec{E} = E_0 z \hat{z}$  through the part of the surface  $z = -3s^2 + 12$  (cylindrical coordinates) that sits above the  $(x, y)$ -plane.

Break this problem into steps:

- Sketch the paraboloid and the vector field.
- Sketch a representative  $d\vec{A}$  and calculate  $d\vec{A}$  algebraically.
- Calculate the flux.

**Solution** We want to compute  $\int \vec{E} \cdot \hat{n} dA$  over the surface of the paraboloid given, or if you prefer (I find it even simpler to think of this way), just  $\int \vec{E} \cdot d\vec{A}$ . We know  $\vec{E}$ , so we really just need to draw little “patches” of area  $d\vec{A}$  on the surface, and figure out what those area vectors look like. That’s not so bad—just draw a little patch somewhere on the surface! Mine is a little square which is  $d\vec{r}_1$  by  $d\vec{r}_2$ . Here I really need a picture, but let’s try to visualize it with words:

Let  $d\vec{r}_1$  point in the  $\hat{\phi}$  direction, with length  $s d\phi$  (Do you see why?) That’s the “bottom” leg of a little patch, which sweeps around in the  $\hat{\phi}$  direction as you change  $\phi$  just a little. It has no  $z$  component, and in fact no radial component either, because this paraboloid sweeps symmetrically around the  $z$ -axis.

The other leg has to be “tilted” up (along) the paraboloid surface. I drew myself a little picture and noticed that this vector points “up the paraboloid” (in order that our cross product will give the outward normal), so it would be written

$$d\vec{r}_2 = ds \hat{s} + dz \hat{z} \quad (1)$$

(Draw it, make sure this makes sense to you!) Do **NOT** add an explicit minus sign on the radial component. The limits of integration will take care of this. Now, “use what you know.”  $d\vec{r}_2$  lives on a parabola, so  $s$  and  $z$  are related by  $z = -3s^2 + 12$ , so  $dz = -6s ds$ . Decide whether you want to integrate with respect to  $s$  or  $z$ . I will choose  $s$ , so I need to change all the  $z$ ’s to  $s$ ’s, (including the  $z$  in the expression for the electric field!) Therefore,  $d\vec{r}_2$  becomes

$$d\vec{r}_2 = (\hat{s} - 6s \hat{z}) ds \quad (2)$$

Now we’re all set.

$$\begin{aligned} d\vec{A} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= s d\phi \hat{\phi} \times (\hat{s} - 6s \hat{z}) ds \\ &= (-s \hat{z} - 6s^2 \hat{s}) ds d\phi \end{aligned} \quad (3)$$

We only CARE about the  $\hat{z}$  component of this expression, because we’re about to dot it with  $\vec{E} = E_0 z \hat{z}$ , and thus we find

$$\vec{E} \cdot d\vec{A} = E_0(-3s^2 + 12)(-s) ds d\phi \quad (4)$$

Almost done! This is our integrand, and we still need to do the double integral. The  $d\phi$  integration is sweet, we sweep around the circle and get  $2\pi$ , since nothing depends on  $\phi$ . So we just have one simple single integration to do, along  $ds$ , which should run from 2 to 0. (Oh, do you see why that is 2? This surface is like a hat, and reaches  $z = 0$  at the point  $s = 2$ .) Make sure to start at  $s = 2$ , the outside of the hat, and run to  $s = 0$  at the top of the hat because that is the direction that  $d\vec{r}_2$  points.

Therefore, the integral that we have to do is:

$$\begin{aligned}\int \vec{E} \cdot d\vec{A} &= \int_2^0 \int_0^{2\pi} E_0(3s^3 - 12s) ds d\phi \\ &= 24\pi E_0\end{aligned}\tag{5}$$

By the way, you can check this answer using the divergence theorem, which we'll talk about later in the week! The surface integral of  $\vec{E} \cdot d\vec{A}$  must equal the volume integral of the divergence of  $E$ , (where  $\vec{\nabla} \cdot \vec{E} = E_0$  in this case). So we just need the volume of the paraboloid. I leave that as an exercise for you (just slice it into circular “plates”); it works! One thing to watch out for is that the surface integral needs to include the flux through the **BOTTOM** of the paraboloid, too. (The divergence theorem only works for **closed** surfaces.) But we're ok, because  $z = 0$  there, and so there is no extra flux from that surface.