The Schrodinger Equation:

$$\hat{H} |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle \tag{1}$$

This equation basically says that if I know how the Hamiltonian transforms a state, I know how the state will evolve with time by solving the differential equation.

At this stage, I'll consider a system where the Hamiltonian doesn't depend on time: $\hat{H} \neq \hat{H}(t)$.

To solve, I'll first write down the time-evolved state in a basis. Since the equation involves the Hamiltonian, I'll use the energy eigenstates as my basis:

$$|\psi(t)\rangle = \sum_m c_m(t) |m\rangle \tag{2}$$

where the energy eigenvalue equation tells me that for any particular $|\phi_m\rangle$:

$$\hat{H} |m\rangle = E_m |m\rangle \tag{3}$$

NOTE: Sometimes, the energy eigenvalue equation is call the time-independent Schrodinger Equation. I'll keep calling it the energy eigenvalue equation.

Notice that the energy eigenstates don't evolve with time. This seems reasonable given that I'm considering a Hamiltonian that doesn't depend on time. Therefore, all the time dependence for the time evolved state is in the probality amplitudes/expansion coefficients $c_m(t)$.

Now, I'll plug this time evolved state written as a sum into the Schrodinger Equation,

$$\hat{H} \sum_m c_m(t) |m\rangle = i\hbar \frac{\partial}{\partial t} \sum_{m'} c_{m'}(t) |m'\rangle \tag{4}$$

where I've used m' to label the terms to distinguish them from the m terms - I don't mean take a derivative.

Now, I'll let the Hamiltonian act on each term in the sum on the left hand side. The expansion coefficients are not affected:

$$\hat{H} \sum_m c_m(t) |m\rangle = \sum_m c_m(t) \hat{H} |m\rangle \tag{5}$$

$$= \sum_m c_m(t) E_m |m\rangle \tag{6}$$

On the right hand side of equation (4), I'll take the time derivative. I can pull the constant and the time derivative inside the sum so that they act on each term. The expansion coefficients depend on time but no other variable. The energy eigenstates do not depend on time.

$$i\hbar \frac{\partial}{\partial t} \sum_{m'} c_{m'}(t) |m'\rangle = \sum_{m'} i\hbar \frac{\partial c_{m'}(t)}{\partial t} |m'\rangle \tag{7}$$

Putting these together in Equation (4), I get:

$$\sum_m c_m(t) E_m \left| m \right\rangle = \sum_m i\hbar \frac{\partial c_{m'}(t)}{\partial t} \left| m' \right\rangle \tag{8}$$

I'm trying to find how the expansion coefficients depend on time, so I need a differential equation in terms of the coefficients. I'll use the orthonormality of the energy eigenstates to find this differential equation. So, I'll multiply from the left by $\langle m''|$.

$$\langle m''| \sum_m c_m(t) E_m \left| m \right\rangle = \langle m''| \sum_m i\hbar \frac{\partial c_{m'}(t)}{\partial t} \left| m' \right\rangle \tag{9}$$

I can bring the $\langle m''|$ into the sums over m and m'.

$$\sum_m c_m(t) E_m \langle m''|m\rangle = \sum_m i\hbar \frac{\partial c_{m'}(t)}{\partial t} \langle m''|m'\rangle \tag{10}$$

Now, I recognize that these brakets are Kronecker Delta's:

$$\langle m''|m\rangle = \delta_{m''m} \tag{11}$$
$$\langle m''|m'\rangle = \delta_{m''m'} \tag{12}$$
$$\tag{13}$$

which transforms all the m's and m''s to m'''s when I perform the sums.

$$\sum_m c_m(t) E_m \delta_{m''m} = \sum_m i\hbar \frac{\partial c_{m'}(t)}{\partial t} \delta_{m''m'} \tag{14}$$
$$c_{m''}(t) E_{m''} = i\hbar \frac{\partial c_{m''}(t)}{\partial t} \tag{15}$$
$$\tag{16}$$

Using m'' is awkward, so I'll switch back to m. Now, I can treat the partial derivative as a total derivative because the expansion coefficient if a function of time only.

$$c_m(t) E_m = i\hbar \frac{dc_m(t)}{dt} \tag{17}$$
$$\tag{18}$$

This equation is now a first order ODE that's separable. First I separate:

$$\frac{dc_m}{c_m} = \frac{-iE_m}{\hbar} dt \tag{19}$$
$$\tag{20}$$

Integrate both sides:

$$\int_{c_m(0)}^{c_m(t)} \frac{dc'_m}{c'_m} = \frac{-iE_m}{\hbar} \int_0^t dt' \tag{21}$$

$$\ln c_m(t) - \ln c_m(0) = \frac{-iE_m}{\hbar} t \tag{22}$$

$$\ln \left[\frac{c_m(t)}{c_m(0)} \right] = \frac{-iE_m}{\hbar} t \tag{23}$$

$$c_m(t) = c_m(0) e^{\frac{-iE_m}{\hbar} t} \tag{24}$$

Plugging this back into equation (2):

$$|\psi(t)\rangle = \sum_m c_m(t) |m\rangle \tag{25}$$

$$|\psi(t)\rangle = \sum_m c_m(0) e^{\frac{-iE_m}{\hbar} t} |m\rangle \tag{26}$$

$$\tag{27}$$

Therefore, to write down the time evolved state for a time-independent Hamiltonian:

1. Find the energy eigenvalues and eigenstates of the Hamiltonian.
2. Write the initial state is the energy eigenstate basis.
3. Multiply each term in the expansion by a complex phase with the corresponding energy eigenvalue

Important features of the time-evolved state:

- You can only put time evolution phases on states that are written in the energy basis
- Each eigenstate gets its own time-dependent phase that includes the energy of that eigenstate
- The time-dependent phase is norm 1 - it's doesn't affect normalization or the relative probabilities associated with energy eigenstates.