

Consider a square wire loop with sides of length L carrying current I . The normal to the plane of the wire loop is at an angle θ with respect to a **uniform** magnetic field $\vec{B}$. Take the direction of the magnetic field to be $\hat{z}$, the origin of coordinates to be at the center of the loop, the high side of the wire to be at constant positive values of x , and the current to be flowing counter-clockwise if looking down along the z -axis.

1. Find the force of the magnetic field on one side of the square loop.

Solution First, I want to draw two good pictures to help me with the geometry here. I'll draw one in the xz -plane and one in the xy -plane, and label each of the sides.

The force is:

$$\vec{F} = \int I d\vec{l} \times \vec{B}$$

For side 1:

$$\begin{aligned}\vec{F}_1 &= \int_{-L/2}^{L/2} I dy \hat{y} \times |\vec{B}| \hat{z} \\ \vec{F}_1 &= IL|\vec{B}|\hat{x}\end{aligned}$$

2. Find the torque on the loop from this force.

Solution On side 1, the moment arm is:

$$\vec{r}_1 = \frac{L}{2} \cos \theta \hat{x} + \frac{L}{2} \sin \theta \hat{z}$$

The torque is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

For side 1:

$$\begin{aligned}\vec{\tau}_1 &= \left[\frac{L}{2} \cos \theta \hat{x} + \frac{L}{2} \sin \theta \hat{z} \right] \times IL|\vec{B}|\hat{x} \\ \vec{\tau}_1 &= \frac{IL^2|\vec{B}|\sin \theta}{2} \hat{y}\end{aligned}$$

3. What is the net force and torque on the loop?

Solution Do the same process for all sides:

$$\begin{aligned}
\vec{F}_1 &= IL|\vec{B}|\hat{x} \\
\vec{F}_2 &= IL|\vec{B}|\hat{y} \\
\vec{F}_3 &= -IL|\vec{B}|\hat{x} \\
\vec{F}_4 &= -IL|\vec{B}|\hat{y} \\
\vec{F}_{net} &= \sum_{n=1}^4 \vec{F}_n = 0
\end{aligned}$$

For calculating the torque, I need all the moment arms:

$$\begin{aligned}
\vec{r}_1 &= \frac{L}{2} \cos \theta \hat{x} + \frac{L}{2} \sin \theta \hat{z} \\
\vec{r}_2 &= L\hat{y} \\
\vec{r}_3 &= -\frac{L}{2} \cos \theta \hat{x} - \frac{L}{2} \sin \theta \hat{z} \\
\vec{r}_4 &= -L\hat{y}
\end{aligned}$$

$$\begin{aligned}
\vec{\tau}_1 &= \frac{IL^2|\vec{B}|\sin \theta}{2} \hat{y} \\
\vec{\tau}_2 &= 0 \\
\vec{\tau}_3 &= \frac{IL^2|\vec{B}|\sin \theta}{2} \hat{y} \\
\vec{\tau}_4 &= 0 \\
\vec{\tau}_{net} &= \sum_{n=1}^4 \vec{\tau}_n = IL^2|\vec{B}|\sin \theta \hat{y}
\end{aligned}$$

Furthermore, I'm noticing that L^2 is the area of the loop. IL^2 is the magnitude of the magnetic moment, $|\vec{\mu}|$. θ is the angle between the normal direction of the loop and the magnetic field. Therefore, I see that:

$$\vec{\tau}_{net} = \vec{\mu} \times \vec{B}$$

- How potential energy is stored in the system by rotating the ring an additional angle θ_f around the y -axis?

Solution

$$\begin{aligned}\Delta U &= -W_{field} \\ &= -\int_{\theta_i}^{\theta_f} \vec{\tau}_{net} \cdot d\vec{\theta} \\ &= -\int_{\theta_i}^{\theta_f} IL^2 |\vec{B}| \sin \theta \hat{y} \cdot d\theta \hat{y} \\ &= -\left(-IL^2 |\vec{B}| \cos \theta \right) \Big|_{\theta_i}^{\theta_f} \\ &= IL^2 |\vec{B}| (\cos \theta_f - \cos \theta_i)\end{aligned}$$

Again, I'm noticing that IL^2 is the magnitude of the magnetic moment, $|\vec{\mu}|$, and this result is the expression I would get if I did the following calculation:

$$U = -\vec{\mu} \cdot \vec{B}$$