Consider a square wire loop with sides of length L carrying current I. The normal to the plane of the wire loop is at an angle θ with respect to a **uniform** magnetic field $\vec{B}$. Take the direction of the magnetic field to be $\hat{z}$, the origin of coordinates to be at the center of the loop, the high side of the wire to be at constant positive values of x, and the current to be flowing counter-clockwise if looking down along the z-axis.

1. Find the force of the magnetic field on one side of the square loop.

Solution First, I want to draw two good pictures to help me with the geometry here. I'll draw one in the xz-plane and one in the xy-plane, and label each of the sides.
The force is:

$$\vec{F} = \int I d\vec{l} \times \vec{B}$$

For side 1:

$$\vec{F_1} = \int_{-L/2}^{L/2} I\, dy\, \hat{y} \times |\vec{B}|\hat{z}$$
$$\vec{F_1} = IL|\vec{B}|\hat{x}$$

2. Find the torque on the loop from this force.

Solution On side 1, the moment arm is:

$$\vec{r_1} = \tfrac{L}{2}\cos\theta\hat{x} + \tfrac{L}{2}\sin\theta\hat{z}$$

.
The torque is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

For side 1:

$$\vec{\tau_1} = \left[\tfrac{L}{2}\cos\theta\hat{x} + \tfrac{L}{2}\sin\theta\hat{z}\right] \times IL|\vec{B}|\hat{x}$$
$$\vec{\tau_1} = \frac{IL^2|\vec{B}|\sin\theta}{2}\,\hat{y}$$

3. What is the net force and torque on the loop?

Solution Do the same process for all sides:

$$\vec{F_1} = IL|\vec{B}|\hat{x}$$
$$\vec{F_2} = IL|\vec{B}|\hat{y}$$
$$\vec{F_3} = -IL|\vec{B}|\hat{x}$$
$$\vec{F_4} = -IL|\vec{B}|\hat{y}$$
$$\vec{F}_{net} = \sum_{n=1}^{4} \vec{F}_n = 0$$

For calculating the torque, I need all the moment arms:

$$\vec{r_1} = \tfrac{L}{2}\cos\theta\hat{x} + \tfrac{L}{2}\sin\theta\hat{z}$$
$$\vec{r_2} = L\hat{y}$$
$$\vec{r_3} = -\tfrac{L}{2}\cos\theta\hat{x} - \tfrac{L}{2}\sin\theta\hat{z}$$
$$\vec{r_2} = -L\hat{y}$$

$$\vec{\tau_1} = \frac{IL^2|\vec{B}|\sin\theta}{2}\,\hat{y}$$
$$\vec{\tau_2} = 0$$
$$\vec{\tau_3} = \frac{IL^2|\vec{B}|\sin\theta}{2}\,\hat{y}$$
$$\vec{\tau_4} = 0$$
$$\vec{\tau}_{net} = \sum_{n=1}^{4} \vec{\tau}_n = IL^2|\vec{B}|\sin\theta\,\hat{y}$$

Furthermore, I'm noticing that L^2 is the area of the loop. IL^2 is the magnitude of the magnetic moment, $|\vec{\mu}|$. θ is the angle between the normal direction of the loop and the magnetic field. Therefore, I see that:

$$\tau_{net} = \vec{\mu} \times \vec{B}$$

4. How potential energy is stored in the system by rotating the ring an additional angle θ_f around the y-axis?

Solution

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\begin{aligned}
\Delta U &= -W_{field} \\
&= -\int_{\theta_i}^{\theta_f} \vec{\tau}_{net} \cdot d\vec{\theta} \\
&= -\int_{\theta_i}^{\theta_f} IL^2|\vec{B}| \sin\theta\, \hat{y} \cdot d\theta\, \hat{y} \\
&= -\Big(-IL^2|\vec{B}| \cos\theta \Big)\Big|_{\theta_i}^{\theta_f} \\
&= IL^2|\vec{B}|(\cos\theta_f - \cos\theta_i)
\end{aligned}
$$

Again, I'm noticing that IL^2 is the magnitude of the magnetic moment, $|\vec{\mu}|$, and this result is the expression I would get if I did the following calculation:

$$U = -\mu \cdot \vec{B}$$