Find the Fourier transform of the (simplified) Gaussian function

$$f(x) = e^{-x^2} \tag{1}$$

You may want to use the value of the following integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{2}$$

Solution

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, e^{-x^2}\, dx \tag{3}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ikx)}\, dx \tag{4}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ikx-k^2/4)-k^2/4}\, dx \tag{5}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-k^2/4} \int_{-\infty}^{\infty} e^{-(x^2+ikx-k^2/4)}\, dx \tag{6}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-k^2/4} \int_{-\infty}^{\infty} e^{-(x+ik/2)^2}\, dx \tag{7}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-k^2/4} \sqrt{\pi} \tag{8}$$

$$= \frac{1}{\sqrt{2}}\, e^{-k^2/4} \tag{9}$$

In eqn (5), we have used an algebra manipulation called Completing the Square. In eqn (7), I have recognized that a shift in the independent variable will not change the value of the integral I was given in the problem statement. If this is not obvious to you, make the substitution $y = x + \frac{ik}{2}$ in the integral in eqn (7).

Notice that the Fourier transform of a Gaussian in the variable x is a Gaussian in the conjugate variable k.