Use the uncertainty relation between position and momentum to estimate the energy of the ground state of a particle in a box (with mass m and length L).

Solution The uncertainty relation between position and momentum is:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

If I minimize this relation and solve for the uncertainty in the momentum:

$$\Delta p = \frac{\hbar}{2\Delta x}$$

For many systems, $\langle p \rangle = 0$, so the square of the uncertainty of *momentum* is equal to the expectation value of the square of the momentum.

$$(\Delta p)^2 = \langle p^2 \rangle - \cancelto{0}{\langle p \rangle^2}$$

For the particle in an infinite square well, maximum uncertainty in the position is L (the particle is in the box).

$$\begin{aligned} E_g &\approx \frac{(\Delta p)^2}{2m} \\ &\approx \left(\frac{\hbar}{2L}\right)^2 \frac{1}{2m} \\ &\approx \frac{\hbar^2}{8mL^2} \end{aligned}$$

Solving for the energy eigenstates exactly yields $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, with a ground state energy of $\frac{\pi^2\hbar^2}{2mL^2}$ Comparing our estimate with the exact solution:

$$\frac{E_{\text{estimate}}}{E_{\text{exact}}} = \frac{\frac{\hbar^2}{8mL^2}}{\frac{\pi^2\hbar^2}{2mL^2}} = \frac{1}{4\pi^2} \approx \frac{1}{36}$$

So, our estimate is 36 times too big, but at least we've recovered that the energy goes like $1/L^2$.