Today we'll be talking about superposition and boundary conditions, which are fundamental in quantum mechanics, but I'll talk for now in the context of musical instruments.

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Figure 1: Coordinates for a guitar string

We've looked at the wave equation for a string under tension (such as a guitar string)

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m/L}\frac{\partial^2 y}{\partial x^2} \tag{1}$$

String boundary conditions The motion of this string is governed by the PDE

$$\frac{\partial^2 y}{\partial x^2} = \frac{m/L}{T}\frac{\partial^2 y}{\partial y^2} \tag{2}$$

For the "A" string of a guitar,

$$\frac{m/L}{T} = 2.5 \times 10^{-5}\ \text{kg/Nm} = 2.5 \times 10^{-5}\frac{\text{s}^2}{\text{m}^2} \tag{3}$$

and we have one additional constraint required to solve this: the boundary conditions that $y = 0$ at each end ($x = 0$ and $x =$ 1m).

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Figure 2: Boundary conditions for a guitar string

Could the string undergo this motion?	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^2 y}{\partial t^2}$	$\frac{\partial^2 y}{\partial x^2}\Big/\frac{\partial^2 y}{\partial t^2}$
$y_1(x,t) = [2\text{ mm}]\sin\left(\frac{\pi}{[1\text{ m}]}x\right)\cos\left(\frac{\pi}{[5\text{ ms}]}t\right)$	$-\frac{\pi^2}{[1\text{ m}]^2}y_1$	$-\frac{\pi^2}{[5\text{ ms}]^2}y_1$	$\frac{[5\times10^{-3}\text{ s}]}{[1\text{ m}]^2} = 2.5\times10^{-5}\frac{\text{s}^2}{\text{m}^2}$
$y_2(x,t) = [2\text{ mm}]\sin\left(\frac{\pi}{[2\text{ m}]}x\right)\cos\left(\frac{\pi}{[10\text{ ms}]}t\right)$			boundary condition!
$y_3(x,t) = [2\text{ mm}]\sin\left(\frac{\pi}{[0.5\text{ m}]}x\right)\cos\left(\frac{\pi}{[2.5\text{ ms}]}t\right)$	$-\frac{\pi^2}{[0.5\text{ m}]^2}y_3$	$-\frac{\pi^2}{[2.5\text{ ms}]^2}y_3$	$\frac{[2.5\times10^{-3}\text{ s}]}{[0.5\text{ m}]^2} = 2.5\times10^{-5}\frac{\text{s}^2}{\text{m}^2}$
$y_4(x,t) = [2\text{ mm}]\sin\left(\frac{\pi}{[0.5\text{ m}]}x\right)\cos\left(\frac{\pi}{[1\text{ ms}]}t\right)$	$-\frac{\pi^2}{[0.5\text{ m}]^2}y_4$	$-\frac{\pi^2}{[1\text{ ms}]^2}y_4$	$\frac{[1\times10^{-3}\text{ s}]}{[0.5\text{ m}]^2} = 1\times10^{-5}\frac{\text{s}^2}{\text{m}^2}$

Superposition If you have two solutions of the wave equation y_1 and y_2, then the sum of them will also be a solution. You can check this by simply plugging $y = y_1 + y_2$ into the wave equation, and using the property that the derivative of a sum is a sum of derivatives.

We can see this happening using a spectrograph, which is a way of visualizing the frequencies present in sound. You can play with this yourself by visiting this spectrogram website which will listen to your microphone and display the intensities of frequencies present. In a spectrograph, thime is on the horizontal axis, and frequency is on the vertical axis.

We can observe the superposition principle in practice by plucking a guitar string in different ways. Usually, when we pluck the string we observe a whole set of harmonics, because the process of plucking gives us a superposition of many sinusoids in the shape of the string. We can also damp out most of the harmonics by lightly touching the string at a node of one of the waves. Or we can pluck the string right in the middle, which omits (mostly) all of the odd modes.

PDE for sound in a gas Wind and brass musical instruments like a trumpet or a bugle (which I don't own) operate by setting up standing waves in a column of air. The differential equation for sound waves in a gas is

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho_{\text{average}}}{\gamma p_{\text{average}}} \frac{\partial^2 p}{\partial t^2} \tag{4}$$

where p is the pressure, ρ is the density of the gas, and p_{average} is the average pressure (typically 1 atmosphere), and γ is the ratio of heat capacities for the gas. You don't really need all those numers, since we can find solutions which looks like

$$p = p_0 \sin(k(x - v_s t)) + [1\ \text{atm}] \qquad v_s = \frac{\omega}{k} = \sqrt{\frac{\gamma p_{\text{average}}}{\rho_{\text{average}}}} \tag{5}$$

$$p = p_0 \sin(kx)\sin(\omega t)) + [1\ \text{atm}] \tag{6}$$

Note that the speed of sound in air is about 340 m/s, which lets us avoid doing math with the pressure and density. Although if you wanted to make predictions for the property of an instrument that is warmed up, you'd have to account both for higher temperature and humidity, both of which affect the speed of sound.

Pipe boundary conditions Brass and wind musical instruments differ in how the boundary conditions are set up. We have two kinds of boundaries, which we call open or closed.

open If the end of a pipe is **open**, then the pressure at that end is essentially fixed at one atmosphere, which puts a node of the wave at that end.

closed If the end of a pipe is **closed**, then the air can't go into our out of that end of the pipe, which makes the pressure form an antinode. This comes because the derivative of the pressure with respect to position is proportional to the velocity of the air, which must be zero at a closed boundary.

Let's start with a pipe open at both ends. Mine is about 34 cm long, which makes our arithmetic easy. Since it is open at each end we have an antinode at each end. What frequencies should we see? Let's look!

FIXME ADD FIGURE WITH WAVE FITTING RIGHT

The lowest-frequency wave with an antinode at each end will fit a half-wavelength into the pipe. Thus

$$\lambda_1 = 2L \tag{7}$$

$$= \frac{v}{f_1} \tag{8}$$

$$f_1 = \frac{v}{2L} \tag{9}$$

$$= \frac{340\ \text{m/s}}{2 \times 34\ \text{m}} \tag{10}$$

$$= 5\ \text{Hz} \tag{11}$$

The next standing wave will have half the wavelength, and then $\frac{1}{3}$ and so on, which gives us 5 Hz, 10 Hz, 15 Hz, ...

In a brass instrument, one end of the pipe is open, and the other is closed. This means that for the fundamental (lowest-frequency) mode the length of the pipe is a quarter of the wavelength of the sound. The next mode that obeys the boundary conditions has the length being three quarters of a wavelength, which gives three times the frequency. Relative to what we saw for the string, there is a missing frequency. And in fact every other frequency is missing.

I don't have a trumpet, but if I close one end of my tube with my hand, I can approximate the same effect. **What frequencies do you expect to see?** Let's try it!

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Our lowest frequency will now be $\frac{1}{4}$ wavelength fitting into the pipe.

$$\lambda_1 = 4L \tag{12}$$
$$= \frac{v}{f_1} \tag{13}$$
$$f_1 = \frac{v}{4L} \tag{14}$$
$$= \frac{340 \text{ m/s}}{4 \times 34 \text{ m}} \tag{15}$$
$$= 2.5 \text{ Hz} \tag{16}$$

The next wave fitting the boundary conditions will have $\frac{3}{4}$ of a wavelength in the pipe, which will thus have three times the fundmental frequency, and the one after will have five times the fundamental.

Flute: your homework.

An oboe has a conical bore which makes things complicated, such that you can't just consider it as a one-dimensional system in the naive way. You might think the same would be true of a trumpet (or other brass instrument), but as it turns out the shape of the bell is designed precisely to make it so the 3D solution gives the same answer as the simple 1D picture, so that it will play in tune.